Characterizing Energy-Related Controllability of Composite Complex Networks via Graph Product

B. She, *Graduate Student Member, IEEE*, S. S. Mehta, E. Doucette, C. Ton, *Member, IEEE*, and Z. Kan, *Member, IEEE*

***Abstract*—This article characterizes the energy-related controllability of composite complex networks. We consider a class of composite networks constructed from simple factor networks via Cartesian product. The considered factor networks are leader-follower signed networks with neighbor-based Laplacian dynamics, adopting positive and negative edges to capture cooperative and competitive interactions among network units. Different from most existing works focusing on classical controllability, this article investigates the controllability of composite networks from energy-related perspectives. Specifically, controllability Gramian-based metrics, including average controllability and volumetric control energy, are characterized based on the Cartesian graph product, which reveals how the energy-related controllability of a composite network can be inferred from the spectral properties of the local factor systems. These results are then extended to layered control networks, a special, yet widely used, network structure in many man-made systems. Since structural balance is a key topological property of signed networks, a necessary and sufficient condition to verify the structural balance of composite signed networks is developed, which is applicable to generalized graph product.**

***Index Terms*—Energy-related network controllability, graph product, signed complex networks.**

I. Introduction

Complex networks can effectively model a variety of natural and man-made systems. Owing to tremendous application potential, growing research has been devoted to investigating structural and functional properties of complex networks. From the viewpoint of advancing design and control of complex networks, a property that is of particular interest to us is the controllability of complex networks, that dictates the ability to steer a network to a desired behavior via external controls.

Manuscript received April 21, 2020; accepted August 8, 2020. Date of publication October 5, 2020; date of current version June 29, 2021. This work was supported in part by the AFRL Mathematical Modeling and Optimization Institute and AFRL Grant FA9453-19- 1-0077. Recommended by Associate Editor A. V. Proskurnikov. *(Corresponding author: Zhen Kan.)*

B. She is with the Department of Mechanical Engineering, The University of Iowa, Iowa City, IA 52242 USA (e-mail: baike-she@uiowa.edu).

S. S. Mehta is with the Department of Mechanical and Aerospace Engineering, University of Florida, Research and Engineering Education Facility, Shalimar, FL 32611 USA (e-mail: siddhart@ufl.edu).

E. Doucette is with the Air Force Research Laboratory, Munitions Directorate, Eglin AFB, FL 32542 USA (e-mail: emily.doucette@eglin.af.mil).

C. Ton is with the Southwest Research Institute, San Antonio, TX 78228 USA (e-mail: chau.t.ton@gmail.com).

Z. Kan is with the Department of Automation, University of Science and Technology of China, Hefei, Anhui 230027, China (e-mail: zkan@ustc.edu.cn).

Color versions of one or more of the figures in this article are available online at https://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2020.3028840

Characterizing the controllability of complex networks is challenging, since complex networks are often constructed of multiple factor systems dynamically interacting among themselves. The factor systems can be basic units such as neurons in a brain network [1], or basic structures consisting of a number of units forming the building blocks of the network such as layered structures in power networks [2]. The dynamics of factors and the interactions among factors, as well as the intrinsic underlying network structures, pose significant challenges in investigating network controllability. In addition, such networks are often large-scale, for instance, with hundreds of drones/robots in a multirobot system, which further complicates the analysis of composite networks.

It has been observed that many complex systems can be constructed and analyzed through simple subsystems (i.e., factors), where core properties of factors are preserved in the composite system. For example, the stability of composite feedback systems can be analyzed based on its factor systems via small-gain theorem and composite Lyapunov functions [3]. Recently, graph products have been explored to construct and reveal structural and functional relationships between factor systems and the associated composite system. In [4]–[7], classical controllability and observability of a composite network were characterized based on its factor networks. In [8], the verification and prediction of the structural balance of signed networks were studied via the Cartesian product. In a recent work [9], generalized graph product, including Cartesian, direct, and strong product, were utilized to reveal the spectral and controllability properties of composite systems. However, the aforementioned results mainly focus on the classical controllability of composite networks.

Besides classical controllability analysis [10]–[18], stemming from practical applicability, an important aspect that needs to be considered is the energy needed for network control. Various metrics have been developed to quantify the energy needed in network control. The properties of the controllability Gramian, such as its minimum eigenvalue [19], the trace of its inverse [20], and the condition number [21], have been explored to characterize the control energy. Representative results include the minimal control energy [22]–[28], energy-constrained controllability [29]–[32], and the joint consideration of network controllability and energy efficiency [33]. Although significant progress has been made in the aforementioned results, energy-related controllability has remained largely uninvestigated for composite complex networks.

This article characterizes the energy-related controllability of composite complex networks. In particular, we consider a class of composite networks constructed from simple factor networks via Cartesian product. The factor network is a leader-follower network with neighbor-based Laplacian feedback, and allows positive and negative edges to capture cooperative and competitive interactions among network units. Due to the graph product, the resulting composite network is a signed leader-follower network. Graph product approaches are leveraged to explore how the energy-related controllability of the composite network can be inferred from its factor systems. Specifically, controllability

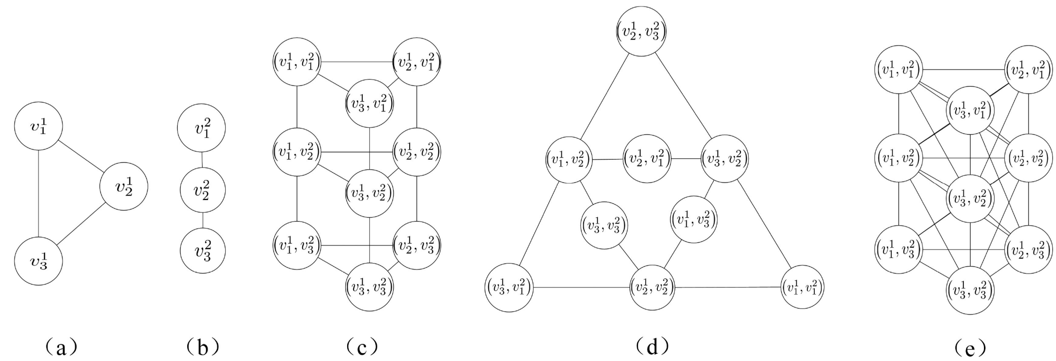

Fig. 1. Factor graphs (a) and (b), and their product graph, (c) $\mathcal{G}_1 \,\square\, \mathcal{G}_2$, (d) $\mathcal{G}_1 \times \mathcal{G}_2$, and (e) $\mathcal{G}_1 \boxtimes \mathcal{G}_2$.

Gramian based metrics, including average controllability and volumetric control energy, are characterized based on the Cartesian graph product, which reveals how the energy-related controllability of a composite network can be inferred from the spectrum of the local factor systems. These results are then extended to layered control networks, a special, yet widely, used network structure in many man-made systems. Since structural balance is a key topological property of signed networks, a necessary and sufficient condition in verifying the structural balance of a signed composite complex network is developed, which is applicable to general graph product.

A crucial benefit of using graph product is that the global properties, such as average controllability and volumetric control energy, of the composite networks can be inferred from its local factor graphs, which provides a practical means to analyze large-scale and complicated networks from its relatively simple factor systems. This article is closely related to [4]. However, the present article focuses on characterizing energy-related measures of network controllability, i.e., the energy needed in network control. Applications, such as network topology design and leader selection, are discussed to demonstrate how factor graphs can be individually designed based on the developed characterizations to improve overall energy-related controllability of the composite system. In addition, signed networks can model a large class of networks with cooperative and competitive interactions among network units, such as social networks and resilient networks [34]. Therefore, the developed energy-related characterizations are not only applicable to competitive networks with possible antagonistic interactions, but also cooperative unsigned networks (i.e., a special case of signed networks with nonnegative edge weights).

II. Preliminaries

Complex networks can be synthesized from a set of smaller size factor graphs via graph product. Consider two undirected graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. Let $\mathcal{G} = \mathcal{G}_1 \triangle \mathcal{G}_2$ denote the composite graph, where $\triangle$ denotes a generalized graph product, including the Cartesian product $\mathcal{G}_1 \,\square\, \mathcal{G}_2$, direct product $\mathcal{G}_1 \times \mathcal{G}_2$, and strong product $\mathcal{G}_1 \boxtimes \mathcal{G}_2$, i.e., $\triangle \in \{\square, \times, \boxtimes\}$. Throughout the rest of this article, the generalized graph product $\triangle$ will be used if the developed result is applicable to any product of $\{\square, \times, \boxtimes\}$, otherwise specific graph product notation will be used. Examples of Cartesian, direct, strong product are illustrated in Fig. 1. Note that these graph products are commutative and associative, i.e., $\mathcal{G}_1 \triangle \mathcal{G}_2$ and $\mathcal{G}_2 \triangle \mathcal{G}_1$ are isomorphic, and $(\mathcal{G}_1 \triangle \mathcal{G}_2) \triangle \mathcal{G}_3$ and $\mathcal{G}_1 \triangle (\mathcal{G}_2 \triangle \mathcal{G}_3)$ are isomorphic for any factor graphs $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$. More details of graph product can be found in [35].

Consider two matrices $A_1 \in \mathbb{R}^{n\times n}$ and $A_2 \in \mathbb{R}^{m\times m}$. The Kronecker product of A_1 and A_2 is denoted by $A_1 \otimes A_2 \in \mathbb{R}^{nm\times nm}$. Further, the Kronecker sum of A_1 and A_2 is defined as $A_1 \oplus A_2 = A_1 \otimes I_m + I_n \otimes A_2$, where I_u is a $u \times u$ identity matrix. The Kronecker product has the following properties: $(A_1 \otimes A_2)(A_3 \otimes A_4) = (A_1 A_3) \otimes (A_2 A_4)$ and $e^{A_1 \oplus A_2} = e^{A_1} \otimes e^{A_2}$. The spectrum of a matrix A is denoted as $\operatorname{eig}(A)$, i.e., the set of eigenvalues of A. Let $A_{i,j}$ denote the (i,j)th entry of A. The ith row and jth column of A are denoted as $A_{i,:}$ and $A_{:,j}$, respectively. The trace and determinant of A are denoted as $\operatorname{tr}(A)$ and $\det A$, respectively.

III. Problem Formulation

A. Leader-Follower Signed Factor Network

Consider a complex network represented by an undirected signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where the node set $\mathcal{V} = \{v_1, \ldots, v_n\}$ and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represent the network units and their interactions, respectively. The interactions are captured by the adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$, where $a_{ij} \neq 0$ if $(v_i, v_j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. No self-loop is considered, i.e., $a_{ii} = 0 \forall i = 1, \ldots, n$. Different from many existing results considering exclusively nonnegative a_{ij} (i.e., an unsigned graph), this article adopts $a_{ij} \in \{\pm 1\}$ to capture cooperative and competitive interactions between network units[1]. Let $d_i = \sum_{j\in\mathcal{N}_i} |a_{ij}|$, where $\mathcal{N}_i = \{v_j | (v_i, v_j) \in \mathcal{E}\}$ denotes the neighbor set of v_i and $|a_{ij}|$ denotes the absolute value of a_{ij}. The graph Laplacian of $\mathcal{G}$ is defined as $\mathcal{L}(\mathcal{G}) \triangleq \mathcal{D} - \mathcal{A}$, where the in-degree matrix $\mathcal{D} \triangleq \operatorname{diag}\{d_1, \ldots, d_n\}$ is a diagonal matrix. Since $\mathcal{G}$ is undirected, the graph Laplacian $\mathcal{L}(\mathcal{G})$ is symmetric. Note that, in unsigned networks, the Laplacian matrix will have negative off-diagonal entries. Hence, the Laplacian of a signed network will, in general, have positive and negative off-diagonal entries, which indicates that zero is no longer a default eigenvalue as in the case of unsigned graphs.

Let $x(t) \in \mathbb{R}^n$ denote the stacked system states. Consider a set $\mathcal{K} = \{v_{l_1}, \ldots, v_{l_m}\} \subseteq \mathcal{V}$ of nodes endowed with external control inputs (i.e., the leaders), where l_i, $i = 1, \ldots m$, indicates the leader's index. Suppose the system states evolve according to the following Laplacian dynamics:

$$\dot{x}(t) = -\mathcal{L}(\mathcal{G})\, x(t) + Bu(t) \tag{1}$$

where $u(t) \in \mathbb{R}^m$ is the external input, and $B = [\, e_{l_1} \cdots e_{l_m} \,] \in \mathbb{R}^{n\times m}$ is the input matrix with basis vector[2] $e_i \in \mathbb{R}^n$, $i = l_1, \ldots, l_m$, indicating the leaders are endowed with external controls. For notational simplicity, $(\mathcal{L}, B)$ will be used throughout this article to represent the dynamics in (1).

If B is well designed such that network controllability is ensured, the system state can be driven from an initial state $x(0) \in \mathbb{R}^n$ to any target

[1]For ease of presentation, $a_{ij} \in \{\pm 1\}$ is used in this article. The developed results still hold if $a_{ij} \in \mathbb{R}$.

[2]A basis vector $e_i \in \mathbb{R}^n$ has zero entries except for the ith entry being one.

state $x_t \in \mathbb{R}^n$ via external input $u(t)$. As indicated in [36], assuming $x(0) = 0$, the minimum control energy required to transit from $x(0)$ to x_f is

$$E(t) = x_f^T \mathcal{W}^{-1}(t) x_f \tag{2}$$

where $\mathcal{W}(t) = \int_0^t e^{-\mathcal{L}\tau} BB^T e^{-\mathcal{L}^T\tau} d\tau$ denotes the controllability Gramian. In this article we focus on the infinite horizon case, i.e., $t \to \infty$, due to the consideration of asymptotic or exponential convergence/stability of dynamic systems.

Since the controllability Gramian provides an energy-related measure of network control, various metrics have been developed based on $\mathcal{W}$. Typical control energy metrics include average controllability tr$(\mathcal{W})$ and volumetric control energy $\log\det\mathcal{W}$ [20]. Due to the trace, tr$(\mathcal{W})$ provides an overall measure of energy expenditure in all directions in the state space. The zero eigenvalues of $\mathcal{W}$ imply uncontrollable directions (i.e., requiring infinite control energy) in the state space. Thus, the smaller the eigenvalue of the inverse of $\mathcal{W}$, the more the control energy needed to transit system states in the corresponding direction. The determinant of $\mathcal{W}$ measures the volume of the ellipsoid containing the target states that can be reached with unit or less control energy. If a system is uncontrollable, the ellipsoid volume is zero, which implies $\det\mathcal{W} = 0$. Since the logarithm is monotone, $\log\det\mathcal{W}$ implies the associated volume in the controllable subspace. Based on tr$(\mathcal{W})$ and $\log\det\mathcal{W}$, the subsequent effort will focus on characterizing energy-related controllability of composite complex networks.

B. Composite Complex System

Direct analysis of energy-related controllability of a composite network can be challenging if the composite network is of large size. Hence, the objective is to characterize the energy-related controllability of the composite system by inferring from its factor systems, taking advantage of the smaller size of the factor systems. As discussed in [4] and [37], a composite network can be decomposed into factor systems in polynomial time.

Without loss of generality, the subsequent development focuses on a composite system $(\mathcal{L}(\mathcal{G}), B)$ constructed by the Cartesian product of two factor systems. The case of two factor systems is adopted for the simplicity of presentation and is not restrictive, since a general composite system with many factor systems can be realized via sequential composition. Particularly, consider two factor systems $(\mathcal{L}(\mathcal{G}_1), B_1)$ and $(\mathcal{L}(\mathcal{G}_2), B_2)$, where $\mathcal{G}_1$ has n nodes with p leaders and $\mathcal{G}_2$ has m nodes with q leaders, i.e., $\mathcal{L}(\mathcal{G}_1) \in \mathbb{R}^{n\times n}$, $\mathcal{L}(\mathcal{G}_2) \in \mathbb{R}^{m\times m}$, $B_1 = [\,e_{l_1^1} \cdots e_{l_p^1}\,] \in \mathbb{R}^{n\times p}$ and $B_2 = [\,e_{l_1^2} \cdots e_{l_q^2}\,] \in \mathbb{R}^{m\times q}$. The basis vectors e_i, $i \in \{l_1^1, \ldots, l_p^1\}$, and e_j, $j \in \{l_1^2, \ldots, l_q^2\}$, indicate the leader nodes v_i and v_j in $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. For notational simplicity, $(\mathcal{L}_1, B_1)$ and $(\mathcal{L}_2, B_2)$ will be used throughout this article to represent the following system dynamics:

$$\dot{x_1}(t) = -\mathcal{L}_1 x_1(t) + B_1 u(t)$$
$$\dot{x_2}(t) = -\mathcal{L}_2 x_1(t) + B_2 u(t)$$

respectively, where $x_1(t) \in \mathbb{R}^n$ and $x_2(t) \in \mathbb{R}^m$ denote the stacked system states. Based on the Cartesian product, the composite dynamics $(\mathcal{L}(\mathcal{G}), B)$ corresponding to $\mathcal{G} = \mathcal{G}_1 \square \mathcal{G}_2$ can be written as

$$\begin{aligned}\dot{x}(t) &= -\mathcal{L}\left(\mathcal{G}\right) x(t) + Bu(t)\\ &= -\left(\mathcal{L}_1 \oplus \mathcal{L}_2\right) x(t) + \left(B_1 \otimes B_2\right) u(t)\end{aligned} \tag{3}$$

where $\mathcal{L}(G_1 \square \mathcal{G}_2) = \mathcal{L}(\mathcal{G}_1) \oplus \mathcal{L}(\mathcal{G}_2)$ is due to the fact that graph Laplacians belong to the family of symmetry preserving representations[3].

IV. Energy-Related Controllability of Cartesian Product Graph

This section focuses on the characterizations of energy-related controllability of composite networks in (3). Specifically, the average controllability and volumetric control energy of $(\mathcal{L}, B)$ are investigated based on its factor systems $(\mathcal{L}_1, B_1)$ and $(\mathcal{L}_2, B_2)$.

Different from unsigned graphs whose graph Laplacian is positive semidefinite by default, when considering signed networks, the graph Laplacian $\mathcal{L}$ can be either positive semidefinite (i.e., $\mathcal{G}$ is structurally balanced) or positive definite (i.e., $\mathcal{G}$ is structurally unbalanced) [34]. To streamline the work, the characterization of the structural balance of the composite graph based on its factor graphs is shown in Section VI. The subsequent development will focus on the cases that $\mathcal{G}$ is structurally unbalanced, i.e., eig$(\mathcal{L})$ contains only positive eigenvalues. If $\mathcal{G}$ is structurally balanced, then, as shown in our recent work [33], a structurally balanced graph can be converted to an unsigned graph under gauge transformation [34], where many existing energy-related characterizations (cf., [38] and [20]) can be immediately applied. In addition, as shown in [30], [39], and[40], the reduced graph Laplacian can be used, where the row and column associated with the zero eigenvalue are removed from the graph Laplacian, so that a structurally balanced graph can be treated as a structurally unbalanced graph via reduced Laplacian matrix when deriving Gramian-based energy metrics. Therefore, we mainly focus on the energy-related characterizations of structurally unbalanced graphs.

A. Characterizations of Average Controllability

Before characterizing the average controllability of the composite system $(\mathcal{L}, B)$ in (3), the following lemma from [41] is introduced.

Lemma 1: Consider two factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ with n and m nodes, respectively. Given the Cartesian product graph $\mathcal{G} = \mathcal{G}_1 \square \mathcal{G}_2$, the graph Laplacian $\mathcal{L}(\mathcal{G})$ takes the form of $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 = \mathcal{L}_1 \otimes I_m + I_n \otimes \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the graph Laplacian of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. The eigenvalues λ_k and eigenvectors u_k of $\mathcal{L}$ are defined as $\lambda_k = \mu_i + \eta_j$ and $u_k = \vartheta_i \otimes w_j$ for $k = 1, \ldots mn$, where (μ_i, ϑ_i), $i = 1, \ldots, n$, and (η_j, w_j), $j = 1, \ldots m$, represent the eigenpairs of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively.

Lemma 1 shows how the eigenpairs of $\mathcal{L}$ can be constructed from the eigenpairs of $\mathcal{L}_1$ and $\mathcal{L}_2$. Let $V = [\vartheta_1 \cdots \vartheta_n]$ and $W = [w_1 \cdots w_m]$ denote the eigenvector matrices of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. We start from the case that each factor system contains a single leader, i.e., $B_1 = b_1 = e_{l_1}$ and $B_2 = b_2 = e_{l_2}$ with $l_1 \in \{1, \ldots, n\}$ and $l_2 \in \{1, \ldots, m\}$, where the basis vectors e_{l_1} and e_{l_2} indicate that v_{l_1} and v_{l_2} are the leaders in $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Based on Lemma 1, the following theorem characterizes the average controllability of $(\mathcal{L}, b_l)$, where $b_l = e_{l_1} \otimes e_{l_2} \in \mathbb{R}^{mn}$ determines the leader v_l in $\mathcal{G}$.

Theorem 1: Provided two factor systems $(\mathcal{L}(\mathcal{G}_1), b_1)$ and $(\mathcal{L}(\mathcal{G}_2), b_2)$, the average controllability of the composite system $(\mathcal{L}(\mathcal{G}_1 \,\square\, \mathcal{G}_2), b_l)$ in (3) can be characterized as

$$\mathrm{tr}\left(\mathcal{W}\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{1}{2\left(\mu_i + \eta_j\right)} V_{l_1,i}^2 W_{l_2,j}^2 \tag{4}$$

where $\mathcal{W}$ is the controllability Gramian of $(\mathcal{L}, b_l)$, μ_i, $i = 1, \ldots, n$, and η_j, $j = 1, \ldots m$, are the spectrum of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, and

[3]A matrix $\mathcal{L}(\mathcal{G})$ is symmetry preserving if, for all permutation $\sigma \in \mathrm{Aut}(\mathcal{G})$, with the corresponding permutation matrix J, $\mathcal{L}(\mathcal{G})J = J\mathcal{L}(\mathcal{G})$.

$V_{l_1,i}$ and $W_{l_2,j}$ represent the (l_1,i)th entry and (l_2,j)th entry of V and W, respectively.

Proof: Consider the composite system $(\mathcal{L}, b_l)$. Since $\mathcal{L}$ is symmetric, it can be written as $\mathcal{L} = U\Lambda U^T \in \mathbb{R}^{mn\times mn}$, where $\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_{mn}\} \in \mathbb{R}^{mn\times mn}$ is a diagonal matrix containing the eigenvalues of $\mathcal{L}$ and $U = [\, u_1 \cdots u_{mn} \,] \in \mathbb{R}^{mn\times mn}$ is the orthogonal eigenvector matrix of $\mathcal{L}$. Based on (3) and using the fact that $e^{-\mathcal{L}\tau} = e^{-U\Lambda U^T\tau} = Ue^{-\Lambda\tau}U^T$, the controllability Gramian can be written as

$$\mathcal{W} = \int_0^\infty e^{-\mathcal{L}\tau} b_l b_l^T e^{-\mathcal{L}\tau} d\tau = U\Gamma U^T \tag{5}$$

where

$$\Gamma = \int_0^\infty e^{-\Lambda\tau} U^T b_l b_l^T U e^{-\Lambda\tau} d\tau. \tag{6}$$

From (6), the ijth entry of Γ is

$$\Gamma_{i,j} = \int_0^\infty e^{-\lambda_i\tau-\lambda_j\tau} U_{l,i}U_{l,j} d\tau = \frac{1}{\lambda_i+\lambda_j} U_{l,i}U_{l,j} \tag{7}$$

where $U_{l,i}$ and $U_{l,j}$ are the (l,i)th and (l,j)th entries of U, respectively.

Since the trace is invariant under cyclic permutations, from (5), the average controllability of $(\mathcal{L}, b_l)$ is

$$\text{tr}\,(\mathcal{W}) = \text{tr}\left(U\Gamma U^T\right) = \text{tr}\left(\Gamma U^T U\right). \tag{8}$$

Substituting (7) into (8) and using Lemma 1

$$\begin{aligned} \text{tr}\,(\mathcal{W}) &= \text{tr}\,(\Gamma) = \sum_{k=1}^{mn} \frac{1}{2\lambda_k} U_{l,k}^2 \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{1}{2\left(\mu_i+\eta_j\right)} V_{l_1,i}^2 W_{l_2,j}^2 \end{aligned} \tag{9}$$

where $V_{l_1,i}$ and $W_{l_2,j}$ are the (l_1,i)th and (l_2,j)th entries of the eigenvector matrices of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. ■

Based on the single leader case in Theorem 1, the following corollary considers multileader cases.

Corollary 1: Consider two factor systems $(\mathcal{L}(\mathcal{G}_1), B_1)$ and $(\mathcal{L}(\mathcal{G}_2), B_2)$, where $\mathcal{G}_1$ has n nodes with p leaders and $\mathcal{G}_2$ has m nodes with q leaders, i.e., $B_1 = [\, e_{l_1^1} \cdots e_{l_p^1} \,] \in \mathbb{R}^{n\times p}$ and $B_2 = [\, e_{l_1^2} \cdots e_{l_q^2} \,] \in \mathbb{R}^{m\times q}$ where the basis vector e_i, $i \in \{l_1^1, \ldots, l_p^1\}$, and e_j, $j \in \{l_1^2, \ldots, l_q^2\}$, indicate the leader nodes v_i and v_j in $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. The average controllability of the composite system $(\mathcal{L}(\mathcal{G}_1 \,\square\, \mathcal{G}_2), B)$ in (3) can be characterized as

$$\text{tr}\,(\mathcal{W}) = \sum_{k=1}^{p}\sum_{l=1}^{q}\sum_{i=1}^{n}\sum_{j=1}^{m} \frac{1}{2\left(\mu_i+\eta_j\right)} V_{l_k^1,i}^2 W_{l_l^2,j}^2$$

where $V_{l_k^1,i}$ and $W_{l_l^2,j}$ are the (l_k^1,i)th entry and (l_l^2,j)th entry of V and W, respectively.

Corollary 1 can be proved following a similar procedure in Theorem 1 and is thus omitted. A key observation from Theorem 1 and Corollary 1 is that, for a Cartesian product composite network, the average controllability can be inferred from the eigenvalues of the factor graph Laplacian and the associated rows of the eigenvector matrices corresponding to the leaders. Hence, Theorem 1 and Corollary1 provide a practical means to characterize tr$(\mathcal{W})$ only based on the eigenvalues and eigenvector matrices of $\mathcal{L}_1$ and $\mathcal{L}_2$, offering a bottom up approach to reveal the properties of a global system from its local systems.

B. Characterizations of Volumetric Control Energy

This section characterizes the volumetric control energy of the composite system $(\mathcal{L}, B)$ based on its factor systems $(\mathcal{L}_1, B_1)$ and $(\mathcal{L}_2, B_2)$. Similarly, results on single leader case is developed first, which are then extended to multiple leaders.

Theorem 2: Consider two factor systems $(\mathcal{L}(\mathcal{G}_1), b_1)$ and $(\mathcal{L}(\mathcal{G}_2), b_2)$ and its corresponding composite system $(\mathcal{L}(\mathcal{G}_1 \,\square\, \mathcal{G}_2), b_l)$ in (3). The volumetric control energy $\log\det\mathcal{W}$ of the composite system can be characterized as

$$\log\det\mathcal{W} = m\log\det\mathcal{W}_1 + n\log\det\mathcal{W}_2 + c$$

where $\mathcal{W}$, $\mathcal{W}_1$, and $\mathcal{W}_2$ are controllability Gramians of the systems $(\mathcal{L}, b_l)$, $(\mathcal{L}_1, b_1)$, and $(\mathcal{L}_2, b_2)$, respectively, and $c = \log\det\overline{\Gamma} - m\log\det\overline{\Gamma}_1 - n\log\det\overline{\Gamma}_2$ is a constant determined by eig$(\mathcal{L}_1)$ and eig$(\mathcal{L}_2)$, with $\overline{\Gamma}_{ij} = \frac{1}{\lambda_i+\lambda_j}$, $[\overline{\Gamma}_1]_{ij} = \frac{1}{\mu_i+\mu_j}$, $[\overline{\Gamma}_2]_{ij} = \frac{1}{\eta_i+\eta_j}$.

Proof: From (5), the volumetric control energy of $(\mathcal{L}, b_l)$ can be written as

$$\begin{aligned} \log\det\mathcal{W} &= \log\left(\det U \det\Gamma \det U^T\right) \\ &= \log\det\Gamma \end{aligned} \tag{10}$$

where $\det U \det U^T = 1$ is used since U is an orthogonal matrix. From (7), Γ can be rewritten as $\Gamma = \overline{U}\overline{\Gamma}\overline{U}$, where $\overline{U} = \text{diag}\{U_{l,1}, U_{l,2}, \ldots, U_{l,mn}\}$ and $\overline{\Gamma} = [\overline{\Gamma}_{ij}] \in \mathbb{R}^{mn\times mn}$ with $\overline{\Gamma}_{ij} = \frac{1}{\lambda_i+\lambda_j}$. Based on $\overline{\Gamma}$, $\overline{U}$, and (10)

$$\begin{aligned} \log\det\mathcal{W} &= \log\left(\det\overline{U}\det\overline{\Gamma}\det\overline{U}\right) \\ &= 2\sum_{i=1}^{mn}\log U_{l,i} + \log\det\overline{\Gamma}. \end{aligned} \tag{11}$$

Expressions, similar to (11), can be obtained for factor systems $(\mathcal{L}_1, b_1)$ and $(\mathcal{L}_2, b_2)$ as

$$\log\det\mathcal{W}_1 = \log\det\left(\overline{V}\overline{\Gamma}_1\overline{V}\right) = 2\sum_{i=1}^{n}\log V_{l_1,i} + \log\det\overline{\Gamma}_1,$$

$$\log\det\mathcal{W}_2 = \log\det\left(\overline{W}\overline{\Gamma}_2\overline{W}\right) = 2\sum_{i=1}^{m}\log W_{l_2,i} + \log\det\overline{\Gamma}_2,$$

where $\overline{V} = \text{diag}\{V_{l_1,1}, V_{l_1,2}, \ldots, V_{l_1,n}\}$, $\overline{\Gamma}_1 \in \mathbb{R}^{n\times n}$ with the ijth entry $(\overline{\Gamma}_1)_{ij} = \frac{1}{\mu_i+\mu_j}$ for $i,j \in \{1,\ldots,n\}$, $\overline{W} = \text{diag}\{W_{l_2,1}, W_{l_2,2}, \ldots, W_{l_2,m}\}$, and $\overline{\Gamma}_2 \in \mathbb{R}^{m\times m}$ with the ijth entry $(\overline{\Gamma}_2)_{ij} = \frac{1}{\eta_i+\eta_j}$ for $i,j \in \{1,\ldots,m\}$.

By Lemma 1, $\overline{U} = \overline{V}\otimes\overline{W}$, and the fact that $\det(\overline{V}\otimes\overline{W}) = (\det\overline{V})^m(\det\overline{W})^n$, $\log\det\mathcal{W}$ in (11) can be written in terms of $\log\det\mathcal{W}_1$ and $\log\det\mathcal{W}_2$ as

$$\begin{aligned} \log\det\mathcal{W} =&\ \log\left(\det\left(\overline{V}\otimes\overline{W}\right)\det\overline{\Gamma}\det\left(\overline{V}\otimes\overline{W}\right)\right) \\ =&\ \log\left(\det\overline{V}\right)^{2\tilde{}m} + \log\left(\det\overline{W}\right)^{2n} + \log\det\overline{\Gamma} \\ =&\ 2m\sum_{j=1}^{n}\log V_{l_1,j} + 2n\sum_{k=1}^{m}\log W_{l_2,k} + \log\det\overline{\Gamma} \\ =&\ m\log\det\mathcal{W}_1 - m\log\det\overline{\Gamma}_1 \\ &+ n\log\det\mathcal{W}_2 - n\log\det\overline{\Gamma}_2 + \log\det\overline{\Gamma} \end{aligned}$$

which completes the proof. ■

Example 1: Consider a multiagent system modeled by an $m \times n$ grid graph. The goal is to identify a leader that enables energy-efficient formation control, i.e., to drive the system to a desired formation with energy considerations. According to Theorem 2, local factor systems can be individually designed for improved volumetric control energy of the composite system. Recall that the grid graph can be decomposed into two path graphs of lengths m and n. Therefore, instead of investigating the volumetric control energy of mn nodes, we only need to investigate the factor path graphs and select the node with the highest volumetric control energy as the leader.

The following theorem considers the case of multiple leaders.

Theorem 3: Consider two factor systems $(\mathcal{L}(\mathcal{G}_1), B_1)$ and $(\mathcal{L}(\mathcal{G}_2), B_2)$, where $\mathcal{G}_1$ has p leaders and $\mathcal{G}_2$ has q leaders, i.e., $B_1 = [\, e_{l_1^1} \cdots e_{l_p^1} \,] \in \mathbb{R}^{n\times p}$ and $B_2 = [\, e_{l_1^2} \cdots e_{l_q^2} \,] \in \mathbb{R}^{m\times q}$. Let $(\mathcal{L}(\mathcal{G}), B)$ be the composite system formed by the Cartesian product of $(\mathcal{L}(\mathcal{G}_1), B_1)$ and $(\mathcal{L}(\mathcal{G}_2), B_2)$, where $B = B_1 \otimes B_2 = [\, e_{l_1} \cdots e_{l_{pq}} \,] \in \mathbb{R}^{nm\times pq}$. The volumetric control energy of the composite system $(\mathcal{L}(\mathcal{G}_1 \,\square\, \mathcal{G}_2), B)$ in (3) can be characterized as

$$\log\det\,(\mathcal{W}) \;= \log\det\left(\hat{V}^T \otimes \hat{W}^T \overline{\Gamma} \hat{V} \otimes \hat{W}\right)$$

where $\hat{V}$, $\hat{W}$, and $\overline{\Gamma}$ are matrices associated with the leaders of $\mathcal{G}$.

Proof: Following a similar proof of Theorem 2, one has $\log\det\mathcal{W} = \log\det(U\Gamma U^T) = \log\det\Gamma$ where $\Gamma = \int_0^\infty e^{-\Lambda\tau} U^T B B^T U e^{-\Lambda\tau} d\tau$ with $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_{mn}\} \in \mathbb{R}^{mn\times mn}$ and $U = [\, u_1 \cdots u_{mn} \,] \in \mathbb{R}^{mn\times mn}$ being the eigenvalue and eigenvector matrix of $\mathcal{L}(\mathcal{G})$, respectively. Similar to (10), Γ can be written as

$$\Gamma = \hat{U}^T \overline{\Gamma} \hat{U} \tag{12}$$

where $\hat{U} = [U_{l_1^1,:}^T \ldots U_{l_{pq}^1,:}^T]^T$ denotes the matrix constructed by the rows of U corresponding to the leaders of $\mathcal{G}$.

Based on Lemma 1, it is evident that $\log\det\mathcal{W}$ of $(\mathcal{L}(\mathcal{G}), B)$ with multiple leaders can be inferred from its factor systems (i.e., the eigenvalues and rows of the eigenvector matrices corresponding to the leaders) by

$$\log\det\left(\hat{U}^T \overline{\Gamma} \hat{U}\right) \;= \log\det\left(\hat{V}^T \otimes \hat{W}^T \overline{\Gamma} \hat{V} \otimes \hat{W}\right) \tag{13}$$

where $\hat{V} = [V_{l_1^1,:}^T \ldots V_{l_p^1,:}^T]^T$ and $\hat{W} = [W_{l_1^2,:}^T \ldots W_{l_q^2,:}^T]^T$ are constructed by the rows of V and W corresponding to the leaders of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. By Lemma 1, the term $\overline{\Gamma}_{ij} = \frac{1}{\lambda_i + \lambda_j}$ can be replaced with $\mathrm{eig}(\mathcal{L}_1)$ and $\mathrm{eig}(\mathcal{L}_2)$, so that the volumetric control energy of $(\mathcal{L}, \mathcal{G})$ can be inferred by the spectral property of $(\mathcal{L}_1, \mathcal{G}_1)$ and $(\mathcal{L}_2, \mathcal{G}_2)$. ■

Similar to Theorem 2, the volumetric control energy can be inferred from the eigenvalues of the factor graph Laplacian and the associated rows of the eigenvector matrices corresponding to the leaders.

V. Energy-Related Controllability of Layered Control Networks

Energy-related measures, i.e., average controllability and volumetric control energy, are characterized in Section IV. This section extends these characterizations to a special, yet widely used, class of networked systems, namely layered control networks. A layered control network is referred to a class of composite networks constructed by the Cartesian graph product, where the composite system has repeated control structure of a factor system. Specifically, consider the same system captured by (3). If the composite input matrix is $B = I_n \otimes B_2$, the form of the input matrix B_2 is repeated in every $\mathcal{G}_2$ layer of the product graph $\mathcal{G}_1 \,\square\, \mathcal{G}_2$. In such networks, each layer shares the same topology of the factor graph $\mathcal{G}_2$ and the layers are connected through the topology of the factor graph $\mathcal{G}_1$. An illustrative example is provided in Fig. 2. Many practical applications feature a layered control network, such as fault detection [42], quantum computing networks [43], and smart sensor networks [44]. Energy-related properties of layered control networks are studied in this section.

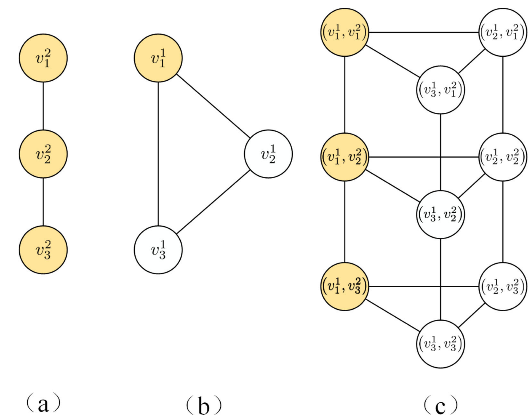

Fig. 2. Example of layered control network, where leaders are marked in yellow. (a) $\mathcal{G}_1$ with $B_1 = I_3$, (b) $\mathcal{G}_2$ with $B_2 = [\,1\ 0\ 0\,]^T$, (c) $\mathcal{G}_1 \,\square\, \mathcal{G}_2$ with $B = I_3 \otimes B_2$.

Similar to the analysis in Section IV, we start with the single leader case. By the definition of layered control network, the dynamics of the composite system $(\mathcal{L}, B)$ is

$$\dot{x}(t) = \;-(\mathcal{L}_1 \oplus \mathcal{L}_2)\, x(t) + (I_n \otimes b_{l_2})\, u(t) \tag{14}$$

where $(\mathcal{L}_1, I_n)$ and $(\mathcal{L}_2, b_{l_2})$ represent the factor systems over $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, and b_{l_2} indicates the v_{l_2} in $\mathcal{G}_2$ is the leader. It is worth pointing out that the following development also applies to the composite system $(\mathcal{L}, B)$ constructed by $(\mathcal{L}_1, b_{l_1})$ and $(\mathcal{L}_2, I_n)$ with $B = b_{l_1} \otimes I_n$, due to the permutation equivalence property of the Kronecker product.

Lemma 2: Consider a composite layered control system $(\mathcal{L}(\mathcal{G}), B)$ with $\mathcal{G} = \mathcal{G}_1 \,\square\, \mathcal{G}_2$ and $B = I_n \otimes b_{l_2}$, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are the factor graphs and b_{l_2} is the input matrix associated with $\mathcal{G}_2$. The controllability Gramian $\mathcal{W}$ of $(\mathcal{L}, B)$ in (3) is

$$\mathcal{W}(t) = (V \otimes W)\, \Psi \left(V^T \otimes W^T\right)$$

where $\Psi = \mathrm{diag}\{\Psi_1, \Psi_2, \ldots, \Psi_n\}$ with $(\Psi_k)_{ij} = \frac{W_{l_2,i} W_{l_2,j}}{2\mu_k + \eta_i + \eta_j}$, $k = 1, \ldots, n$, $i, j = 1, \ldots, m$, where μ_k and η_i are the eigenvalues and $V \in \mathbb{R}^{n\times n}$ and $W \in \mathbb{R}^{m\times m}$ are the eigenvector matrices of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively.

Proof: Based on (5)

$$\begin{aligned}\mathcal{W} &= \int_0^\infty e^{-\mathcal{L}\tau} B B^T e^{-\mathcal{L}\tau} d\tau \\ &= U \left(\int_0^\infty e^{-\Lambda\tau} U^T B B^T U e^{-\Lambda\tau} d\tau\right) U^T d\tau \end{aligned} \tag{15}$$

where U and Λ are the eigenvector and eigenvalue matrices of $\mathcal{L}$, respectively. Since $U = V \otimes W$, (15) can be expanded as

$$\begin{aligned}\mathcal{W} = V \otimes W \Bigg(&\int_0^\infty e^{-\Lambda\tau} \left(V^T \otimes W^T\right) (I_n \otimes b_{l_2}) \\ &\cdot \left(I_n^T \otimes b_{l_2}^T\right) (V \otimes W)\, e^{-\Lambda\tau} d\tau \Bigg) V^T \otimes W^T\end{aligned}$$

$$= V \otimes W \left(\int_0^\infty e^{-\Lambda\tau} \left(V^T I_n I_n^T V\right) \right.$$
$$\left. \otimes \left(W^T b_{l_2} b_{l_2}^T W\right) e^{-\Lambda\tau} d\tau \right) V^T \otimes W^T$$
$$= (V \otimes W) \Psi \left(V^T \otimes W^T\right)$$

where

$$\Psi = \int_0^\infty e^{-\Lambda\tau} I_n \otimes \left(W^T b_{l_2} b_{l_2}^T W\right) e^{-\Lambda\tau} d\tau.$$

Define $\overline{\Psi} = \mathrm{diag}\{\overline{\Psi}_1, \overline{\Psi}_2, \ldots, \overline{\Psi}_n\} = I_n \otimes (W^T b_{l_2} b_{l_2}^T W) \in \mathbb{R}^{mn\times mn}$, which is a block diagonal matrix with $\overline{\Psi}_i = W_{l_2,:}^T W_{l_2,:} \in \mathbb{R}^{m\times m}$ for $i = 1, \ldots, n$, where $W_{l_2,:} \in \mathbb{R}^m$ is the l_2th row of W. Thus, $\Psi = \mathrm{diag}\{\Psi_1, \Psi_2, \ldots, \Psi_n\} \in \mathbb{R}^{mn\times mn}$ is also a block diagonal matrix with $\Psi_i = \int_0^\infty e^{-\Lambda_i\tau}\overline{\Psi}_i e^{-\Lambda_i\tau} d\tau \in \mathbb{R}^{m\times m}$ for $i = 1, \ldots, n$, where Λ is rewritten as $\Lambda = \mathrm{diag}\{\Lambda_1, \Lambda_2, \ldots, \Lambda_n\}$ with the diagonal entries of $\Lambda_i \in \mathbb{R}^{m\times m}$ indicating the $((i-1)m+1)$th to kmth eigenvalues of $\mathcal{L}$. Let λ_i^k denote the ith diagonal entry of $\boldsymbol{\Lambda}_k$. Then, the entries of the kth block Ψ_k can be represented as

$$(\Psi_k)_{ij} = \frac{W_{l_2,i} W_{l_2,j}}{\lambda_i^k + \lambda_j^k} = \frac{W_{l_2,i} W_{l_2,j}}{2\mu_k + \eta_i + \eta_j} \tag{16}$$

where $\lambda_i^k = \mu_k + \eta_i$ and $\lambda_j^k = \mu_k + \eta_j$ from Lemma 1 are used, where μ_k, $k = 1, \ldots n$ is the eigenvalue of $\mathcal{L}_1$ while η_i and η_j for $i, j = 1, \ldots, m$ are the eigenvalues of $\mathcal{L}_2$. ■

Based on the controllability Gramian developed in Lemma 2, energy-related measures are characterized in the following theorem.

Theorem 4: Consider a composite layered control system $(\mathcal{L}(\mathcal{G}_1 \,\square\, \mathcal{G}_2), B)$ with $B = I_n \otimes b_{l_2}$, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are the factor graphs and b_{l_2} is the input matrix associated with $\mathcal{G}_2$ with v_{l_2} being the leader. The average controllability of $(\mathcal{L}, B)$ is characterized as $\mathrm{tr}(\mathcal{W}) = \sum_{k=1}^n \sum_{j=1}^m \frac{W_{l_2,j}^2}{2(\mu_k+\eta_j)}$, and the volumetric control energy of $(\mathcal{L}, B)$ is characterized as

$$\log\det(\mathcal{W}) = n \log\det \mathcal{W}_2 + c_1$$

where $c_1 = -n \log\det \overline{\Gamma}_2 + \sum_{k=1}^n \log\det(\Gamma_k)$.

The proof of Theorem 4 is omitted, since it can be obtained using Lemma 2 and following a similar proof as in Theorem 1 and Theorem 2. Theorem 4 indicates that the energy-related controllability of the composite layered network can be determined by the eigenvalues of the factor systems (i.e., $\mathrm{eig}(\mathcal{L}_1)$ and $\mathrm{eig}(\mathcal{L}_2)$) and the l_2th row of W corresponding to the leader node v_{l_2} in $\mathcal{G}_2$.

Theorem 4 can be trivially extended to multileader cases. To see that, consider a composite layered control system $(\mathcal{L}(\mathcal{G}), B)$ with the input matrix $B = I_n \otimes B_2$, where $B_2 = [\, e_{l_1^2} \cdots e_{l_{m_2}^2} \,] \in \mathbb{R}^{m\times m_2}$ indicates $\mathcal{G}_2$ contains a set of m_2 leaders indexed by $l_i^2, i = 1, \ldots, m_2$. Based on Lemma 2, the controllability Gramian of $(\mathcal{L}(\mathcal{G}), B)$ with multiple leaders can be written as

$$\mathcal{W} = (V \otimes W)\, \widetilde{\Psi} \left(V^T \otimes W^T\right) \tag{17}$$

where $\widetilde{\Psi} = \int_0^\infty e^{-\Lambda\tau} I_n \otimes (W^T B_2 B_2^T W) e^{-\Lambda\tau} d\tau$. Similar to the proof of Lemma 2, $\widetilde{\Psi} = \mathrm{diag}\{\widetilde{\Psi}_1, \widetilde{\Psi}_2, \ldots, \widetilde{\Psi}_n\}$ is a diagonal matrix with ijth entry of each block $\widetilde{\Psi}_k$, $k = 1, \ldots, n$, defined as

$$\left(\widetilde{\Psi}_k\right)_{ij} = \frac{W_{l_1^2,i} W_{l_1^2,j} + W_{l_2^2,i} W_{l_2^2,j} + \cdots + W_{l_{m_2}^2,i} W_{l_{m_2}^2,j}}{2\mu_k + \eta_i + \eta_j}. \tag{18}$$

Since $\mathrm{tr}(\mathcal{W})$ and $\log\det(\mathcal{W})$ are defined based on $\mathcal{W}$ given in (17) and (18), following similar analysis as in Theorem 1, it is straightforward to show that the $\mathrm{tr}(\mathcal{W})$ and $\log\det(\mathcal{W})$ of $(\mathcal{L}(\mathcal{G}), B)$ can be inferred from the eigenvalues and the eigenvector matrices of the factor systems.

VI. Structural Balance of General Product Graphs

Other than the Cartesian product, composite systems can also be constructed based on direct and strong product of factor systems. Due to the importance of structural balance, this section develops a necessary and sufficient condition to characterize the structural balance of signed composite networks by its factor networks, which is generally applicable to Cartesian, direct, and strong graph product.

Consider a graph represented by general product graph $\mathcal{G} = \mathcal{G}_1 \triangle \mathcal{G}_2$, where $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{A}_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{A}_2)$ are factor graphs. A necessary and sufficient condition to characterize structural balance is provided below.

Proposition 1: [34] A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is structurally balanced if and only if there exists a gauge transformation matrix $\Phi = \mathrm{diag}\{\sigma_1, \ldots, \sigma_n\}$ with $\sigma_i \in \{\pm 1\}$ such that $\Phi\mathcal{A}\Phi$ has nonnegative entries.

Proposition 1 indicates that structural balance of $\mathcal{G}$ can be characterized based on its adjacency matrix $\mathcal{A}$. When considering generalized graph product $\triangle$, as indicated in [9]

$$\begin{aligned}\mathcal{A} &= \mathcal{A}\left(\mathcal{G}_1 \triangle \mathcal{G}_2\right) \\ &= \alpha_1 \mathcal{A}_1 \otimes I_m + \alpha_2 I_n \otimes \mathcal{A}_2 + \alpha_3 \mathcal{A}_1 \otimes \mathcal{A}_2\end{aligned} \tag{19}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \{0, 1\}$. Different graph products can be realized through different combinations of α_i for $i = 1, 2, 3$. For instance, the adjacency matrix $\mathcal{A}$ of the Cartesian product $\square$, direct product $\times$, and strong product $\boxtimes$, can be realized if $[\alpha_1, \alpha_2, \alpha_3]$ take the value of $[1, 1, 0]$, $[0, 0, 1]$, and $[1, 1, 1]$, respectively.[4] Although structural balance has proven to be invariant under the Cartesian product $\mathcal{G}_1 \,\square\, \mathcal{G}_2$ in [8], Theorem 5 extends the result in [8] to the generalized product graph $\mathcal{G}_1 \triangle \mathcal{G}_2$ via gauge transformation.

Theorem 5: Consider two signed graphs $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{A}_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{A}_2)$. The generalized product graph $\mathcal{G} = \mathcal{G}_1 \triangle \mathcal{G}_2 = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is structurally balanced if and only if $\mathcal{G}_1$ and $\mathcal{G}_2$ are structurally balanced.

Proof: To obtain the necessary condition, suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are two structurally balanced signed graphs. According to Proposition 1, there exist gauge transformation matrices Φ_1 and Φ_2 such that $\Phi_1\mathcal{A}_1\Phi_1$ and $\Phi_2\mathcal{A}_2\Phi_2$ have nonnegative entries. Defining a diagonal matrix $\Phi = \Phi_1 \otimes \Phi_2$

$$\begin{aligned}\Phi\mathcal{A}\Phi &= (\Phi_1 \otimes \Phi_2)\left(\alpha_1\mathcal{A}_1 \otimes I_m + \alpha_2 I_n \otimes \mathcal{A}_2\right. \\ &\quad \left. + \alpha_3 \mathcal{A}_1 \otimes \mathcal{A}_2\right)(\Phi_1 \otimes \Phi_2) \\ &= \alpha_1 \Phi_1\mathcal{A}_1\Phi_1 \otimes I_m + \alpha_2 I_n \otimes \Phi_2\mathcal{A}_2\Phi_2, \\ &\quad + \alpha_3 \left(\Phi_1\mathcal{A}_1\Phi_1\right) \otimes \left(\Phi_2\mathcal{A}_2\Phi_2\right)\end{aligned}$$

where the fact that $\Phi_1\Phi_1 = I_n$ and $\Phi_2\Phi_2 = I_m$ are used. Since $\alpha_1, \alpha_2, \alpha_2 \in \{0, 1\}$, it can be shown that $\alpha_1\Phi_1\mathcal{A}_1\Phi_1 \otimes I_m$, $\alpha_2 I_n \otimes \Phi_2\mathcal{A}_2\Phi_2$, and $\alpha_3(\Phi_1\mathcal{A}_1\Phi_1) \otimes (\Phi_2\mathcal{A}_2\Phi_2)$ have nonnegative entries. Therefore $\Phi\mathcal{A}\Phi$ is structurally balanced with the gauge transformation matrix Φ by Proposition 1.

To show the sufficient condition, suppose the product graph $\mathcal{G}$ is structurally balanced. By proposition 1, there exists a gauge transformation matrix Φ such that $\Phi\mathcal{A}\Phi$ has nonnegative entries. Assuming

[4]Other product graphs such as skew product and converse skew product can be obtained when the vector takes the value $[0, 1, 1]$ and $[1, 0, 1]$, respectively.

that $\Phi = \Phi_1 \otimes \Phi_2$ yields

$$\begin{aligned}\Phi\mathcal{A}\Phi &= \alpha_1 \Phi\mathcal{A}_1 \otimes I_m\Phi + \alpha_2\Phi I_n \otimes \mathcal{A}_2\Phi + \alpha_3\Phi\mathcal{A}_1 \otimes \mathcal{A}_2\Phi \\ &= \alpha_1\Phi_1\mathcal{A}_1\Phi_1 \otimes I_m + \alpha_2 I_n \otimes \Phi_2\mathcal{A}_2\Phi_2, \\ &\quad + \alpha_3 (\Phi_1\mathcal{A}_1\Phi_1) \otimes (\Phi_2\mathcal{A}_2\Phi_2)\end{aligned}$$

where Φ_1 and Φ_2 are diagonal matrices with with ± 1 as the diagonal entries. It can be verified that, if the (i, j)th entry of one of $\alpha_1\Phi_1\mathcal{A}_1\Phi_1 \otimes I_m$, $I_n \otimes \Phi_2\mathcal{A}_2\Phi_2$ and $(\Phi_1\mathcal{A}_1\Phi_1) \otimes (\Phi_2\mathcal{A}_2\Phi_2)$ is nonzero, the (i, j)th entry of the other two matrices must be zero. In other words, the nonzero entries of $\Phi\mathcal{A}\Phi$ are determined by the corresponding entries either from $\Phi_1\mathcal{A}_1\Phi_1 \otimes I_m$,$I_n \otimes \Phi_2\mathcal{A}_2\Phi_2$, or $(\Phi_1\mathcal{A}_1\Phi_1) \otimes (\Phi_2\mathcal{A}_2\Phi_2)$. If $\Phi\mathcal{A}\Phi$ only has nonnegative entries, the entries of $\Phi_1\mathcal{A}_1\Phi_1 \otimes I_m, I_n \otimes \Phi_2\mathcal{A}_2\Phi_2, (\Phi_1\mathcal{A}_1\Phi_1) \otimes (\Phi_2\mathcal{A}_2\Phi_2)$ must be all nonnegative, indicating Φ_1 and Φ_2 are the gauge transformation matrices of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Therefore, both $\mathcal{G}_1$and $\mathcal{G}_2$ are structurally balanced by Proposition 1. ■

Remark 1: In the case of direct product, it is possible that the product graph $\mathcal{G}$ becomes unconnected even if the factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are connected. For such cases, Theorem 5 applies to each connected component of $\mathcal{G}$, i.e., each connected component is structurally balanced if and only if the factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are structurally balanced. The proof is straightforward by reorganizing $\Phi\mathcal{A}\Phi$ into a block diagonal matrix via graph automorphism and thus omitted here.

Theorem 5 indicates that the structural balance of the composite network $\mathcal{G}$ can be efficiently determined or constructed by the structural balance properties of its factor graphs. Although only two factor graphs are considered, Theorem 5 can be extended for the case of multiple factor graphs obviously.

VII. Conclusion

Energy-related controllability measures, i.e., average controllability and volumetric control energy, are characterized via graph product approaches in this article. Since the present article is mainly developed for undirected composite networks, additional research will consider extending current results to directed composite networks.

Acknowledgment

Any opinions, findings and conclusions or recommendations expressed in this article are those of the author(s) and do not necessarily reflect the views of the funding agency.

References

[1] D. S. Bassett and E. Bullmore, "Small-world brain networks," *Neuroscientist*, vol. 12, no. 6, pp. 512–523, 2006.
[2] Y. Zhang, L. Wang, W. Sun, R. C. Green II, and M. Alam, "Distributed intrusion detection system in a multi-layer network architecture of smart grids," *IEEE Trans. Smart Grid*, vol. 2, no. 4, pp. 796–808, Dec. 2011.
[3] W. M. Haddad and V. Chellaboina, *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*. Princeton, NJ, USA: Princeton Univ. Press, 2011.
[4] A. Chapman, M. Nabi-Abdolyousefi, and M. Mesbahi, "Controllability and observability of network-of-networks via Cartesian products," *IEEE Trans. Autom. Control*, vol. 59, no. 10, pp. 2668–2679, Oct. 2014.
[5] G. Notarstefano and G. Parlangeli, "Controllability and observability of grid graphs via reduction and symmetries," *IEEE Trans. Autom. Control*, vol. 58, no. 7, pp. 1719–1731, Jul. 2013.
[6] Y. Hao, Q. Wang, Z. Duan, and G. Chen, "Controllability of Kronecker product networks," *Automatica*, vol. 110, 2019, Art. no. 108597.
[7] Y. Hao, Z. Duan, G. Chen, and F. Wu, "New controllability conditions for networked identical LTI systems," *IEEE Trans. Autom. Control*, vol. 64, no. 10, pp. 4223–4228, Oct. 2019.
[8] L. Pan, H. Shao, and M. Mesbahi, "Verification and prediction of structural balance: A data-driven perspective," in *Proc. IEEE, Amer. Control Conf.*, 2016, pp. 2858–2863.
[9] T. N. Tran and A. Chapman, "Generalized graph product: Spectrum, trajectories and controllability," in *Proc. IEEE Conf. Decis. Control.*, 2018, pp. 5358–5363.
[10] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt, "Controllability of multi-agent systems from a graph-theoretic perspective," *SIAM J. Control Optim*, vol. 48, no. 1, pp. 162–186, 2009.
[11] A. Yazıcıoğlu, W. Abbas, and M. Egerstedt, "Graph distances and controllability of networks," *IEEE Trans. Autom. Control*, vol. 61, no. 12, pp. 4125–4130, Dec. 2016.
[12] B. She, S. Mehta, C. Ton, and Z. Kan, "Controllability ensured leader group selection on signed multi-agent networks," *IEEE Trans. Cybern.*, vol. 50, no. 1, pp. 222–232, Jan. 2020.
[13] C. O. Aguilar and B. Gharesifard, "Graph controllability classes for the Laplacian leader-follower dynamics," *IEEE Trans. Autom. Control*, vol. 60, no. 6, pp. 1611–1623, Jun. 2015.
[14] C. Commault and J.-M. Dion, "Input addition and leader selection for the controllability of graph-based systems," *Automatica*, vol. 49, no. 11, pp. 3322–3328, 2013.
[15] B. She and Z. Kan, "Characterizing controllable subspace and herdability of signed weighted networks via graph partition," *Automatica*, vol. 115, pp. 1–7, 2020.
[16] X. Liu, H. Lin, and B. M. Chen, "Structural controllability of switched linear systems," *Automatica*, vol. 49, no. 12, pp. 3531–3537, 2013.
[17] O. Romero and S. Pequito, "Actuator placement for symmetric structural controllability with heterogeneous costs," *IEEE Control Syst. Lett.*, vol. 2, no. 4, pp. 821–826, Oct. 2018.
[18] M. K. Mehrabadi, M. Zamani, and Z. Chen, "Structural controllability of a consensus network with multiple leaders," *IEEE Trans. Autom. Control*, vol. 64, no. 12, pp. 5101–5107, Dec. 2019.
[19] G. Yan, J. Ren, Y.-C. Lai, C.-H. Lai, and B. Li, "Controlling complex networks: How much energy is needed?" *Phys. Rev. Lett.*, vol. 108, no. 21, 2012 , Art. no. 218703.
[20] T. H. Summers, F. L. Cortesi, and J. Lygeros, "On submodularity and controllability in complex dynamical networks," *IEEE Trans. Control Netw. Syst.*, vol. 3, no. 1, pp. 91–101, 2016.
[21] F. Pasqualetti, S. Zampieri, and F. Bullo, "Controllability metrics, limitations and algorithms for complex networks," *IEEE Trans. Control Netw. Syst.*, vol. 1, no. 1, pp. 40–52, 2014.
[22] V. Tzoumas, M. A. Rahimian, G. J. Pappas, and A. Jadbabaie, "Minimal actuator placement with bounds on control effort," *IEEE Trans. Control Netw. Syst.*, vol. 3, no. 1, pp. 67–78, Mar. 2016.
[23] S. Pequito, S. Kar, and A. P. Aguiar, "Minimum cost input/output design for large-scale linear structural systems," *Automatica*, vol. 68, pp. 384–391, 2016.
[24] J. Ding, C. Wen, and G. Li, "Key node selection in minimum-cost control of complex networks," *Physica A*, vol. 486, pp. 251–261, 2017.
[25] G. Li, P. Tang, C. Wen, and Z. Meng, "Boundary constraints for minimum cost control of directed networks," *IEEE Trans. Cybern.*, vol. 47, no. 12, pp. 4196–4207, 2017.
[26] A. Clark, B. Alomair, L. Bushnell, and R. Poovendran, "Input selection for performance and controllability of structured linear descriptor systems," *SIAM J. Control Optim.*, vol. 55, no. 1, pp. 457–485, 2017.
[27] A. Olshevsky, "Eigenvalue clustering, control energy, and logarithmic capacity," *Syst. Control Lett.*, vol. 96, pp. 45–50, 2016.
[28] G. Lindmark and C. Altafini, "Minimum energy control for complex networks," *Sci. Rep.*, vol. 8, no. 1, 2018, Art. no. 3188.
[29] G. Yan, G. Tsekenis, B. Barzel, J.-J. Slotine, Y.-Y. Liu, and A.-L. Barabási, "Spectrum of controlling and observing complex networks," *Nature Phys.*, vol. 11, no. 9, pp. 779–786, 2015.
[30] K. Fitch and N. E. Leonard, "Joint centrality distinguishes optimal leaders in noisy networks," *IEEE Trans. Control Netw. Syst.*, vol. 3, no. 4, pp. 366–378, 2016.
[31] C. O. Becker, S. Pequito, G. J. Pappas, and V. M. Preciado, "Network design for controllability metrics," in *Proc. IEEE Conf. Decis. Control*, 2017, pp. 4193–4198.
[32] N. Bof, G. Baggio, and S. Zampieri, "On the role of network centrality in the controllability of complex networks," *IEEE Trans. Control Netw. Syst.*, vol. 4, no. 3, pp. 643–653, Sep. 2017.
[33] B. She and Z. Kan, "Algebraictopological characterizations of structural balance in signed graphs," *Automatica*, vol. 107, pp. 61–67, 2019.
[34] C. Altafini, "Consensus problems on networks with antagonistic interactions," *IEEE Trans. Autom. Control*, vol. 58, no. 4, pp. 935–946, Apr. 2013.

[35] W. Imrich and S. Klavzar, *Product Graphs: Structure and Recognition*. Hoboken, NJ, USA: Wiley, 2000.

[36] T. Kailath, *Linear Systems*, vol. 156. Englewood Cliffs, NJ: Prentice-Hall, 1980.

[37] J. Feigenbaum, “Directed graphs have unique cartesian factorizations that can be found in polynomial time,” *Discrete Appl. Math.*, vol. 15, pp. 105–110, 1986.

[38] A. Olshevsky, “On (non) supermodularity of average control energy,” *IEEE Trans. Control Netw. Syst.*, vol. 5, no. 3, pp. 1177–1181, Sep. 2018.

[39] I. Poulakakis, G. F. Young, L. Scardovi, and N. E. Leonard, “Information centrality and ordering of nodes for accuracy in noisy decision-making networks,” *IEEE Trans. Autom. Control*, vol. 61, no. 4, pp. 1040–1045, Apr. 2016.

[40] R. Dhal and S. Roy, “Vulnerability of network synchronization processes: A minimum energy perspective,” *IEEE Trans. Autom. Control*, vol. 61, no. 9, pp. 2525–2530, Sep. 2016.

[41] M. Fiedler, “Algebraic connectivity of graphs,” *Czechoslovak Math. J.*, vol. 23, no. 2, pp. 298–305, 1973.

[42] J. Leskovec *et al.*, “Cost-effective outbreak detection in networks,” in *Proc. ACM SIGKDD Int. Conf. Knowl. Discovery Data Mining*, ACM, 2007, pp. 420–429.

[43] D. Burgarth, D. D’Alessandro, L. Hogben, S. Severini, and M. Young, “Zero forcing, linear and quantum controllability for systems evolving on networks,” *IEEE Trans. Autom. Control*, vol. 58, no. 9, pp. 2349–2354, Sep. 2013.

[44] V. Gupta, M. Sharma, and N. Thakur, “Optimization criteria for optimal placement of piezoelectric sensors and actuators on a smart structure: A technical review,” *J. Intell. Mater. Syst. Struct.*, vol. 21, no. 12, pp. 1227–1243, 2010.